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Q1 An acute angled triangle $\mathcal{T}$ is inscribed in circle Ω.Denote by Γ the nine-point circle of $\mathcal{T}$.A circle ω passes through two of the vertices of $\mathcal{T}$, and centre of Ω.Prove that the common external tangents of Γ and ω meet on the external bisector of the angle at third vertex of $\mathcal{T}$.

Q2 Determine all non-constant monic polynomials $P(x)$ with integer coefficients such that no prime $p > 10^{100}$ divides any number of the form $P(2^n)$

Q3 Let ABC be a triangle with I as incenter.The incircle touches BC at D.Let D' be the antipode of D on the incircle.Make a tangent at D' to incircle.Let it meet (ABC) at X, Y respectively.Let the other tangent from X meet the other tangent from Y at Z.Prove that (ZBD) meets IB at the midpoint of IB

Q4 Let n be a fixed positive integer.
- Show that there exist real polynomials $p_1, p_2, p_3, \cdots, p_k \in \mathbb{R}[x_1, \cdots, x_n]$ such that

$$(x_1+x_2+\cdots+x_n)^2+p_1(x_1,\cdots,x_n)^2+p_2(x_1,\cdots,x_n)^2+\cdots+p_k(x_1,\cdots,x_n)^2 = n(x_1^2+x_2^2+\cdots+x_n^2)$$

- Find the least natural number k, depending on n, such that the above polynomials $p_1, p_2, \cdots, p_k$ exist.

Q5 Sheldon was really annoying Leonard. So to keep him quiet, Leonard decided to do something. He gave Sheldon the following grid

1	1	1	1	1	0
1	1	1	1	0	0
1	1	1	0	0	0
1	1	0	0	0	1
1	0	0	0	1	0
0	0	0	1	0	0

and asked him to transform it to the new grid below

1	2	18	24	28	30
21	3	4	16	22	26
23	19	5	6	14	20
32	25	17	7	8	12
33	34	27	15	9	10
35	31	36	29	13	11

by only applying the following algorithm:

• At each step, Sheldon must choose either two rows or two columns.

• For two columns c_1, c_2, if a, b are entries in c_1, c_2 respectively, then we say that a and b are corresponding if they belong to the same row. Similarly we define corresponding entries of two rows. So for Sheldon's choice, if two corresponding entries have the same parity, he should do nothing to them, but if they have different parities, he should add 1 to both of them.

Leonard hoped this would keep Sheldon occupied for some time, but Sheldon immediately said, "But this is impossible!". Was Sheldon right? Justify.
