**TESSELLATE PRESENTS**

# STEMS

Scholastic Test of Excellence in Mathematical Sciences

## Mathematics Category A

**Exam Date : 26th January 2020**
**Exam Timing : 12noon - 6pm**

# Rules and Regulations

## Marking Scheme

1. The question paper is divided in two parts -
   Objective (14 questions) + Subjective (6 questions).
2. Each objective question is worth **2 points**.
3. Each subjective problem is worth **10 points**.
4. There is no negative marking in any section.
5. **The subjective part will be graded only if you score above a certain cut-off (to be decided later) in the objective section of the paper**.
6. **For the final score, your total score (subjective + objective) will be taken into consideration.**

## Solution guidelines

1. You are **NOT** required to show your work for the objective part of the paper.
   Only tick your option choices in a tabular format drawn on a blank sheet.
   A sample is shown below.

| Q. No. | (a) | (b) | (c) | (d) |
|---|---|---|---|---|
| 1 | | ✓ | | |
| 2 | | | ✓ | |
| 3 | | ✓ | | |
| 4 | ✓ | | | |
| 5 | | | | ✓ |

Fig. - Sample objective answer submission

2. Provide a complete solution/proof for subjective questions. The solutions must be correct, original, detailed and clearly understandable for full credit.
   Partial credit might be awarded to incomplete proofs, based on the progress made towards solving the problem.
3. Draw clear, well-labeled diagrams wherever necessary.

## Miscellaneous

1. Answers should be your own and should reflect your independent thinking process.
2. Do **NOT** post the questions on any forums or discussion groups. It will result in immediate disqualification of involved candidates when caught.
3. Answers should be written clearly, in a legible way. Formal proofs are required wherever asked for. Unclear reasoning might not be awarded points, draw clear diagrams wherever necessary.
4. Sharing/discussion aimed towards solving or distribution of problems appearing in the contest while the contest is live in any kind of online platform/forum shall be considered as a failure in complying with the regulations.
5. Any form of plagiarism or failure to comply with aforementioned regulations may lead to disqualification.

## Contact details - ONLY for subject related queries

- Please do not call these people for technical problems or submission inquiries. Only if you find an ambiguity in a question and need clarification, use these contacts.
- As our phone numbers will be busy, **we prefer WhatsApp & email queries** instead. Only call us if absolutely necessary.
- Try to solve all your submission related doubts from information in the next page ONLY. We have included all details in the next section.

| | |
|---|---|
| Saketh Narayanan - 9611034545 | Lakshay Modi - 9811503488 |
| Query email - stems.2020.enq.math@gmail.com | |

- **Do NOT call any number to ask if your submission has reached us.** **If you send your submission to the right email address with mentioned details, we will receive it. We will contact all participants who fail to submit, so please be patient.**

# How to submit your answers

1. Write the following details **as per your registration** on the first page of your submission file/photographs -

| | | |
|---|---|---|
| Name | : | Your full name |
| School/College name | : | |
| Class/Year of Study | : | Class 8, Class 11, Undergrad 1st year etc. |
| Registered Email address | : | |
| Mode of S.T.E.M.S. Registration | : | Online / Through School / Other (mention details) |

2. Write all your answers on sheets of paper, following all the solution guidelines. Write the page number in the top right corner of every sheet.

3. Scan your answers or take clear photographs of your response sheets. Compile them into a single PDF file or send all pictures in the right sequence.

   If you have a limited file size issue when sending your submission, make a new Google Drive folder with title '**Your Name - Mathematics A submission**'.

4. Send the submission file or Google Drive link from your registered email -

| | | |
|---|---|---|
| Submission email address | : | **stems.2020.math.a@gmail.com** |
| Subject of email | : | Mathematics Category A Submission - STEMS |
| Submission deadline | : | 26th January 2020 - 18:00 |

Good luck, happy problem-solving!

# Objective Questions

For **Problems 1-6**, each problem has **four** options, namely **(a), (b), (c), (d)**, of which ***only one*** is correct, **2 point** will be awarded for correctly answering a problem, **NO** negative marks shall be awarded for wrong answers/unattempted problems .

For **Problems 7-14**, there are **NO** options provided. The solution to each problen is either a number, expression or sentence. **NO** proof of your solution is required.

**Problem 1.** *Find the number of integral solutions,* $(x, y)$ *to the following equation:*
$xy(x + y + 1) = 2019^{2020} + 1$
*a. 1*
*b. 0*
*c.* 2019
*d.* 2020

**Problem 2.** *Let* $f : \mathbb{N}\cup\{0\} \to \mathbb{N}\cup\{0\}$ *be such that* $f(0, x) = f(x, 0) = 1$ *for all* $x \in \mathbb{N}\cup\{0\}$*. Also for all* $i, j \in \mathbb{N}$ *we have* $f(i, j) = f(i - 1, j - 1) + f(i, j - 1)$*. Find* $f(1009, 2019)$*.*
*a.* $2^{2018}$
*b.* $2^{2019}$
*c.* $2^{2021}$
*d.* $2^{2020}$

**Problem 3.** *Let* $*$ *be a binary operation on* $\mathbb{Z}^2$ *defined as follows:*
$(a, b) * (c, d) = (ac - 2bd, bc + ad)$*.*
*Find the number of solutions to the equation:* $(a, b) * (c, d) = (3, 7)$
*a. 32*
*b.* $\infty$
*c. 16*
*d. 0*

**Problem 4.** *Let* $a, b \in \mathbb{N}$ *with* $gcd(a, b) = 1$*. Find the number of ordered pairs* $(a, b)$ *such that:*
$\frac{a}{b} + \frac{201b}{10201a} \in \mathbb{N}$
*a. 1*
*b.* $\infty$
*c. 0*
*d. 2*

**Problem 5.** $X_1, X_2 \cdots X_{13}$ *is regular* 13 *gon inscribed in a circle* $\Gamma$. *Let* $A_0 = X_1, B_0 = X_2, C_0 = X_4$. *Given* $A_iB_iC_i$, *we construct* $A_{i+1}B_{i+1}C_{i+1}$ *in following way:* $A_i + 1$ *is the meeting point of the altitude of* $A_i$ *in triangle* $A_iB_iC_i$ *,extended as a line and* $\Gamma . B_{i+1} and C_{i+1}$ *are defined in a similar way.*
*Find minimum the* $i > 0$ *such that* $A_i = A_0, B_i = B_0, C_i = C_0$.
*a. 7*
*b. 6*
*c. 13*
*d. 12*

**Problem 6.** 2020 *points are placed in a circle. We colour each vertex with one of 5 colours. What's the maximum number of chords you are guaranteed to draw (regardless of the coloring of the points) in the circle such that the endpoints of the chords have same color and no two chord intersect (not even at the boundary of the circle) ?*
*a. 404*
*b. 203*
*c. 336*
*d. 403*

**Problem 7.** *There are 3 piles of coins with* 2020 *coins each. A and B play a game, taking turns to play their moves. A move consists of either taking one coin from each pile (that is,* $(a, b, c) \mapsto (a-1, b-1, c-1)$*), or choosing two piles and removing as much as coins one wants, but at least one from those two piles (i.e.* $(a, b, c) \mapsto (a, b-x, c-y)$ *with* $x + y \geq 1$ *and cyclic variants). The player who removes the last coin wins. A goes first. Who has a winning strategy?*

**Problem 8.** *Let* $ABC$ *be a triangle inscribed in circle* $\Gamma$ *with radius* 1. *Let* $M, N$ *be the feet of internal bisectors of angles* $B$ *and* $C$. *Line* $MN$ *meets* $\Gamma$ *again at* $P$ *and* $Q$, *and lines* $BM$ *and* $CN$ *meet at* $I$. *Determine the radius of the circumcircle of triangle* $IPQ$

**Problem 9.** *In triangle* $ABC$ *with* $\angle A = 60°$, $AB = 4, BC = 8$, *let* $\omega$ *denote the nine-point circle. Let* $AX, AY$ *be tangents from* $A$ *to* $\omega$. *Determine* $XY^2$.
*For information about the nine-point circle, refer to the following link:*
`https://en.wikipedia.org/wiki/Nine-point_circle`

**Problem 10.** *There are* $m$ *trolls and* $n$ *dwarves on the left side of a river. They have one boat to reach the opposite side of the river. The boat can fit any number of trolls and/or dwarves and they can use it as many times they want to reach to the opposite side, but a troll/dwarf can row the boat back from right side to left side only once. Also two trolls can't be on the boat at any time. Determine all pairs* $(m, n)$ *with* $0 \leq m, n \leq 2020$ *for which everyone can reach the opposite side.*

**Problem 11.** *In triangle* $ABC$, *the incircle* $\omega$ *touches sides* $BC, CA, AB$ *at* $D, E, F$ *respectively. Let* $P$ *be the foot of perpendicular from* $D$ *to* $EF$. *Let* $AB = 13, BC = 14, CA = 15$.

*Determine $BP : PC$.*

**Problem 12.** *Let $ABC$ be a triangle with $AB = 4, AC = 9$. Let the external bisector of angle $A$ meet the circumcircle of triangle $ABC$ again at $M \neq A$. A circle with center $M$ and radius $MB$ meets the internal bisector of angle $A$ at points $P$ and $Q$. Determine the length of $PQ$.*

**Problem 13.** *Let $ABC$ be a triangle circumcircle $\Gamma$ and incircle $\omega$. Let $\Omega$ be the circle tangent to $\Gamma$ and the sides $AB$, $BC$, and let $X = \Gamma \cap \Omega$. Let $Y$, $Z$ be distinct points on $\Gamma$ such that $XY$, $YZ$ are tangent to $\omega$. Suppose radius of $\omega$ is 1, and radius of $\Gamma$ is $R$, and angle $BAC = 60^\circ$. Determine $YZ^2$ in terms of $R$.*

**Problem 14.** *Let $ABC$ be a triangle inscribed in circle $\Gamma$. Let $AB = 2, AC = 3, BC = 4$ and $D, E$ be points on rays $AB$ and $AC$ such that $AD = 6, AE = 4$. Let $F$ be the midpoint of $DE$ and line $AF$ meet $\Gamma$ at $G \neq A$. Determine the ratio $[GAB] : [GAC]$.*

# Subjective Problems

**Problem 1.** *Given are four positive reals* $b_1, b_2, b_3, b_4$ *such that* $b_1^2 + b_3^2 = b_2^2 + b_4^2$. *Positive reals* $a_1, a_2, a_3, a_4$ *satisfy* $a_i^2 + a_{i+1}^2 = b_i^2$ *for* $i = 1, 2, 3, 4$ *where indices are taken mod 4. Express the maximium value that* $(a_1 + a_3)(a_2 + a_4)$ *can take in terms of* $b_1, b_2, b_3, b_4$.

**Problem 2.** *How many triangles with integer sides and largest side* $n$ *are there? If two triangles are congruent, they are considered to be same.*

**Problem 3.** $AB$ *and* $CD$ *are parallel line segments.* $AC$ *meets* $BD$ *at* $F$ *in such a way that* $F$ *is inside* $ABCD$. $E$ *is on segment* $AB$. $P$ *is on* $EC$ *such that* $\angle BPC = \angle BEF$. $Q$ *on* $ED$ *such that* $\angle AQD = \angle AEF$. *Prove that* $PQCD$ *is cyclic.*

**Problem 4.** *In triangle* $ABC$ *with incenter* $I$, *the incircle* $\omega$ *touches sides* $\overline{AC}$ *and* $\overline{AB}$ *at points* $E$ *and* $F$, *respectively. A circle passing through* $B$ *and* $C$ *touches* $\omega$ *at point* $K$. *The circumcircle of* $\triangle KEC$ *meets* $\overline{BC}$ *at* $Q \neq C$. *Prove that* $\overline{FQ} \parallel \overline{BI}$

**Problem 5.** *Find the number of solutions(in* $\mathbb{N}$*) to the following equation:*
$\phi(n^4 + 1) = 8n$
***note***: $\phi$ *is the euler totient function (*$\phi(n)$ *is the number of natural numbers less than* $n$ *that are co-prime to* $n$*.)*

**Problem 6.** *Find all positive integers* $n$ *for which we can select* $2^{n-1}$ *distinct subsets* $S_1, \cdots, S_{2^{n-1}}$ *of* $\{1, 2, \cdots, n\}$ *satisfying the following condition:*
*for distinct* $1 \leq i, j \leq 2^{n-1}, S_i \cap S_j \neq \phi \implies |i - j| \geq 4$