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Q1 **Black hole thermodynamics**

The goal of this problem is to explore some interesting properties of Black Holes. The following equation was obtained by L. Smarr in 1973:

$$M^2 = \frac{1}{16\pi}A + \frac{4\pi}{A}\left(J^2 + \frac{1}{4}Q^4\right) + \frac{1}{2}Q^2$$

where M, J, Q and A are the mass, angular momentum, charge and area of the event horizon of a black hole.

To make contact with thermodynamics we write for the entropy of the Black Hole,

$$S = \frac{1}{4}k_B A$$

where k_B is the Boltzmann constant.

- Work in natural units $G = \hbar = c = 1$ and show that the equation for the entropy is dimensionally correct.
- Take $k_B = 1/8\pi$ (by choosing units) and derive an expression for $S(M, J, Q)$. Is this expression unique? (Hint: What is the entropy of the Schwarzschild Black Hole which corresponds to $J = Q = 0$?)

We suppose the mass-energy M (since $c = 1$) plays the role of internal energy. Show that T, Ω, Φ defined via,

$$dM = TdS + \Omega dJ + \Phi dQ$$

are given by,

$$T = \tfrac{1}{M}\left[1 - \tfrac{1}{16S^2}\left(J^2 + \tfrac{1}{4}Q^4\right)\right]$$
$$\Omega = \tfrac{J}{8MS}$$
$$\Phi = \tfrac{Q}{2M}\left[1 + \tfrac{Q^2}{8S}\right].$$

This is the analog of the first law of thermodynamics.
-Look at the expression for $M(S, J, Q)$ closely and derive the analog of the Gibbs-Duhem Relation familiar from Thermodynamics.
- Show that,

$$S \to \frac{1}{4}M^2 - \frac{1}{8}Q^2$$

as $T \to 0$. What does this say about the third law of thermodynamics? Give reasons to support your answer.

An alternative statement to the third law is that ”it is impossible to reach absolute-zero in a finite number of steps”. What can we conclude from part (e)?

Q2 **Little Mario and the Cylindrical Beam**
Little Mario wishes to jump over a very long (practically infinite) cylindrical beam of radius r whose axis is at a height h from the ground. With what minimum initial speed must he launch himself if:

- Mario is allowed to touch the beam (neglect frictional effects)?

- Mario is not allowed to touch the beam?

Approximate Little Mario by a point particle for convenience. Acceleration due to gravity is g.

Q3 **Classical Probability Distribution for Quantum States?**
The goal of this problem is to try and mimic a Statistical Mechanics approach to Quantum Mechanics. In Classical Statistical Mechanics one has the usual Gibbs-Boltzmann Formula which gives the probability distribution in phase-space to be:

$$\rho(x_1, \dots, x_n, p_1, \dots, p_n) \sim \exp(-\beta H(x_1, \dots, x_n, p_1, \dots, p_n))$$

where H is the Hamiltonian of the system.

- Why can’t we demand a similar probability distribution over phase-space in Quantum Mechanics?

If the wave function $\psi(x_1, \dots, x_n)$ is given, we construct the following expression:

$$\begin{aligned} &P(x_1, \dots, x_n, p_1, \dots, p_n) \\ &= \left(\frac{1}{\pi\hbar}\right)^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dy_1 \dots dy_n \psi^*(x_1 + y_1, \dots, x_n + y_n) \\ &\times \psi(x_1 - y_1, \dots, x_n - y_n) \exp\left(\frac{2i}{\hbar}(p_1 y_1 + \dots + p_n y_n)\right) \end{aligned}$$

- Show that,

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dp_1 \dots dp_n P(x_1, \dots, x_n, p_1, \dots, p_n) = |\psi(x_1, \dots, x_n)|^2$$

which are the correct probabilities for the co-ordinates.

- Show that,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \ldots dx_n \, P(x_1, \ldots, x_n, p_1, \ldots, p_n) = \left| \tilde{\psi}(p_1, \ldots, p_n) \right|^2$$

which are the correct probabilities for the momenta where,

$$\tilde{\psi}(p_1, \ldots, p_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \ldots dx_n \psi(x_1, \ldots, x_n) \exp\left(-\frac{i}{\hbar}(x_1 p_1 + \cdots + x_n p_n)\right)$$

is the Fourier transform of the wave-function $\psi(x_1, \ldots, x_n)$.

- The function P defined above therefore seems to be a good candidate for a probability distribution in Quantum Mechanics. Would this not contradict part (a)? Give reasons to support your answer.