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Q1 We are given k colors and we have to assign a single color to every vertex. An edge is **satisfied** if the vertices on that edge, are of different colors.

-Prove that you can always find an algorithm which assigns colors to vertices so that at least $\frac{k-1}{k}|E|$ edges are satisfied where $|E|$ is the cardinality of the edges in the graph.

-Prove that there is a poly time deterministic algorithm for this

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Q2 Given two forests A and B with $V(A) = V(B)$, that is the graphs are over same vertex set. Suppose A has **strictly more** edges than B. Prove that there exists an edge of A which if included in the edge set of B, then B will still remain a forest. Graphs are undirected

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Q3 Let Σ be a finite set. For $x, y \in \Sigma^*$, define

$$x \preceq y$$

if x is a sub-string (**not necessarily contiguous**) of y. For example, $ac \preceq abc$. We call a set $S \subseteq \Sigma^*$ **good** if $\forall x, y \in \Sigma^*$,

$$x \preceq y,\ y \in S \quad \Rightarrow \quad x \in S.$$

Prove or disprove: Every good set is regular.

Q4 A set M of natural numbers is called a *spectrum* if there is a first-order language L and a sentence ϕ over L such that:

$$M = \{n \mid \phi \text{ has a model containing exactly } n \text{ elements}\}$$

For example, consider a sentence $\phi = \exists e.(\forall x.x = e)$ in a first order language with no relation symbol, no function symbol, and no constant symbol. The formula ϕ only admits a model containing exactly 1 element. Therefore, the set $\{1\}$ is a spectrum.

Show that:

- Every finite subset of $\mathbb{N} \setminus \{0\}$ is a spectrum
- The set of even numbers, i.e., $\{2k \mid k \in \mathbb{N}\}$ is a spectrum
- For any fixed $m \geq 1$, the set of numbers greater than 0 that are divisible by m, i.e., $\{m \cdot k \mid k \in \mathbb{N}\}$ is a spectrum

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Q5 Let's say a language $L \subseteq \{0,1\}^*$ is in $\mathbf{P}_{angel}$ if there exists a polynomial $p : \mathbb{N} \mapsto \mathbb{N}$, a sequence of strings $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n \in \{0,1\}^{p(n)}$, and a deterministic polynomial time Turing Machine M such that for every $x \in \{0,1\}^n$

$$x \in L \Leftrightarrow M(x, \alpha_n) = 1$$

Let us call α_n to be the *angel string* for all x of the length n. Note that the *angel string* is **not** similar to a *witness* or *certificate* as used in the definition of **NP** For example, all unary languages, even $UHALT$ which is undecidable, are in $\mathbf{P}_{angel}$ because the *angel string* can simply be a single bit that tells us if the given unary string is in $UHALT$ or not.

A set $S \subseteq \Sigma^*$ is said to be **sparse** if there exists a polynomial $p : \mathbb{N} \mapsto \mathbb{N}$ such that for each $n \in \mathbb{N}$, the number of strings of length n in S is bounded by $p(n)$. In other words, $|S^{=n}| \leq p(n)$, where $S^{=n} \subseteq S$ contains all the strings in S that are of length n.

- Given $k \in \mathbb{N}$ sparse sets $S_1, S_2 \ldots S_k$, show that there exists a sparse set S and a deterministic polynomial time TM M with oracle access to S such that given an input $\langle x, i\rangle$ the TM M will accept it if and only if $x \in S_i$.

Define the set S (note that it need not be computable), and give the description of M with oracle S.

Note that a TM M with oracle access to S can query whether $s \in S$ and get the correct answer in return in constant time.

- Let us define a variant of $\mathbf{P}_{angel}$ called $\mathbf{P}_{bad-angel}$ with a constraint that there should exists a polynomial time algorithm that can **compute** the angel string for any length $n \in \mathbb{N}$. In other words, there is a poly-time algorithm A such that $\alpha_n = A(n)$.

Is $\mathbf{P} = \mathbf{P}_{bad-angel}$? Is $\mathbf{NP} = \mathbf{P}_{bad-angel}$? Justify.

- Let the language $L \in \mathbf{P}_{angel}$. Show that there exists a sparse set S_L and a deterministic polynomial time TM M with oracle access to S_L that can decide the language L.